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# Classes of Commuting Matrices

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**Abstract.** Sets of square matrices of order $n$ having special forms are defined as $n_k$-classes. Conditions for these $n_k$-classes to be commutative rings are stated. Some examples illustrate the presented theory.

**Keywords:** Commuting matrices; Matrix equations; Upper triangular Toeplitz matrices.

## 1. Introduction

Let $A$ be a real matrix. To obtain all real matrices that commute with $A$ is equivalent to find all solutions of the matrix equation

$$AX = XA. \tag{1}$$

First we consider two particular cases.

We suppose

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

these kind of matrices form a field and they commute with all matrices, that is any matrix $X$ of order 2 is solution of (1).

We now consider

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

we will see later that the matrices

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$$X = \begin{bmatrix} x_1 & x_2 \\ 0 & x_1 \end{bmatrix}$$

for $x_1, x_2 \in \mathbb{R}$ are solutions of (1) and the set of all matrices with this form is a commutative ring.

Returning now to the equation (1), a classical approach to solve this equation is to consider the Jordan canonical form $J_A$ of $A$, that is $A = SJ_AS^{-1}$, where $S$ is the respective nonsingular similarity matrix [1].

So, the equation (1) is equivalent to the equation

$$J_AY = YJ_A, \tag{2}$$

in which $Y = S^{-1}XS$.

Thus, $X$ is a solution of the equation (1) if and only if $Y$ is a solution of the equation (2). We see this in the following example.

*Example 1.1.* Let

$$A = \begin{bmatrix} 11 & -2 & 0 \\ 8 & 1 & 0 \\ -4 & 1 & 6 \end{bmatrix},$$

whose Jordan canonical form and similarity matrix are

$$J_A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix},$$

respectively. Solving the equation (2) we obtain the following solution set

$$\left\{ Y = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Hence,

$$\left\{ X = SYS^{-1} = \begin{bmatrix} \frac{1}{3}(-x_1 + 4x_3) & \frac{1}{3}(x_1 - x_3) & 0 \\ -\frac{4}{3}(x_1 - x_3) & \frac{1}{3}(4x_1 - x_3) & 0 \\ \frac{4}{3}(x_2 - x_3) & \frac{1}{3}(x_3 - x_2) & x_2 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\}$$

is the solution set of the equation (1).

## 2. Basic Theory

In this Section, we have the basic facts about the solution sets of equations (1) and (2). We also see that the triangular Toeplitz matrices play an important role in such theory.

**Definition 2.1.** *An upper triangular Toeplitz matrix of order n is*

$$T_n = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ 0 & x_1 & x_2 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & x_3 \\ \vdots & \ddots & \ddots & x_1 & x_2 \\ 0 & \dots & 0 & 0 & x_1 \end{bmatrix}, x_i \in \mathbb{R}.$$

We also define the sets

$$\mathcal{T}_n = \{T_n\},$$

$$\mathcal{T}_n^+ = \{T_n, x_1 \neq 0\},$$

$$\mathcal{N}_n = \{T_n, x_1 = 0\},$$

$$\mathcal{H}_n = \{T_n, x_1 = 0, x_2 = 1, x_3 = x_4 = \ldots = x_n = 0\}.$$

We observe that $\mathcal{T}_n = \mathcal{T}_n^+ \cup \mathcal{N}_n, \mathcal{T}_n^+ \cap \mathcal{N}_n = \emptyset$ and that any $N_n \in \mathcal{N}_n$ and any $H_n \in \mathcal{H}_n$ are nilpotent, considering that $H_n^n = 0$ and $N_n^n = 0$. Furthermore, $H_n^k$ is an upper triangular Toeplitz matrix which all elements are 0 except in the $(k+1)st$ superdiagonal where the elements are 1.

Furthermore, every upper triangular Toeplitz matrix $T_n \in \mathcal{T}_n$ can be represented as a linear combination of powers of the matrix $H_n \in \mathcal{H}_n$:

$$T_n = x_1 I + x_2 H_n + x_3 H_n^2 + \ldots + x_n H_n^{n-1} = \sum_{i=0}^{n-1} x_{i+1} H_n^i. \tag{3}$$

Using (3) the next results are easily verify.

**Proposition 2.2.** *The set $\mathcal{T}_n$ is a commutative ring.*

**Proposition 2.3.** *The set $\mathcal{T}_n^+$ is a commutative ring with unity.*

**Proposition 2.4.** *The set $\mathcal{N}_n$ is a commutative ring with no unity.*

We consider now the set

$$\mathcal{K}_n = \left\{ diag\left( K_{n_1}, K_{n_2}, \ldots, K_{n_p} \right), \sum_{i=1}^{p} n_i = n \right\},$$

where $n_i$ are the order of the blocks $K_{n_i}$ and $K_{n_i} \in \mathcal{T}_n^+$ or $K_{n_i} \in \mathcal{N}_n$.

As a direct consequence of Propositions 2.3 e 2.4, we have:

**Proposition 2.5.** *$\mathcal{K}_n$ is a commutative ring.*

*Remark 2.6.* Given a set $\mathcal{K}'_n$, if for all $i$, $K'_{n_i} \in \mathcal{T}_n^+$, then $\mathcal{K}'_n$ has unity equal to $I_n$. On the other hand, given a set $\mathcal{K}''_n$, if for any $j$, we have that $K_{n_j} \notin \mathcal{T}_n^+$ only if $K''_{n_j} = 0 \in \mathbb{R}$, then $\mathcal{K}''_n$ has unity different of $I_n$.

*Example 2.7.* Let

$$\mathcal{K}_4 = \left\{ \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\},$$

so we have that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the unity of $\mathcal{K}_4$.

Next, we see how the properties of Jordan canonical form of a matrix affect the solution set of the equation (1).

**Definition 2.8.** *[5] An $m \times n$ matrix $C$ is called triangularly striped if*

1. *For $m = n$*
$$C = T_n,$$
2. *For $m < n$, if $l = n - m$*
$$C = \begin{bmatrix} 0_l & T_m \end{bmatrix},$$
3. *For $m > n$, if $l = n - m$*
$$C = \begin{bmatrix} T_n \\ 0_l \end{bmatrix}.$$

**Theorem 2.9.** *[4], [5] Let $A$ be a real matrix. If the Jordan canonical form of $A$ is*

$$J_A = diag\left( J_{n_1}(\lambda), J_{n_2}(\lambda), \ldots, J_{n_k}(\lambda) \right),$$

*in which $\lambda$ is the unique eigenvalue of $A$ and $n_1, n_2, \ldots, n_k$ are the partial multiplicities of $\lambda$, with $\sum_{i=1}^{k} n_i = n$ then, the solution set of the equation (2) is the set*

$$\mathcal{Z}_n = \left\{ \begin{bmatrix} Z_{11} & Z_{12} & \ldots & Z_{1k} \\ Z_{21} & Z_{22} & \ldots & Z_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{k1} & Z_{k2} & \ldots & Z_{kk} \end{bmatrix} \right\},$$

*where* $Z_{ij},\ i,j = 1,\ldots,k$ *is an arbitrary triangularly striped matrix of order* $n_i \times n_j$*.*

**Corollary 2.10.** *[4], [5] For* $n_1 \geq n_2 \geq \ldots \geq n_k$*, the number of arbitrary parameters in the solution set of the equation* (2) *is given by*

$$n_1 + 3n_2 + 5n_3 + \ldots + (2k-1)n_k.$$

We now extend it to a general matrix having $p$ eigenvalues.

**Corollary 2.11.** *If a real matrix* $A$ *has* $p$ *distinct eigenvalues then*

($i$) *the solution set of the equation* (2) *is the set*

$$\mathcal{W} = \left\{ diag\left( W_1, W_2, \ldots, W_p \right) \right\}$$

*where* $W_1, W_2, \ldots, W_p$ *are matrices of the same form of the matrices of* $\mathcal{Z}_n$ *,*

($ii$) *the number of arbitrary parameters in the solution set is*

$$\sum_{i=1}^{p} (n_{1_i} + 3n_{2_i} + \ldots + (2k_i - 1)n_{k_i}),$$

*where* $n_{j_i}$*, for* $j = 1,\ldots,k$*, are the partial multiplicities of the eigenvalues* $\lambda_i$*, for* $i = 1,\ldots,p$*.*

The nonderogatory matrices, that is, the matrices whose Jordan form has only one Jordan block for each different eigenvalue, have a key role in our work. Next we have an important fact for such matrices, which is a direct consequence of Theorem 2.9.

**Corollary 2.12.** *Let* $A$ *be a nonderogatory real matrix with* $p$ *disctint eigenvalues and respective* $n_1, n_2, \ldots, n_p$ *multiplicities, then*

($i$) *the solution set of the equation* (2) *is the set*

$$\mathcal{B}_n = \left\{ B = diag\left( B_{n_1}, B_{n_2}, \ldots, B_{n_p} \right), \sum_{i=1}^{p} n_i = n \right\},$$

*where* $B_{n_1}, B_{n_2}, \ldots, B_{n_p} \in \mathcal{T}_n$*,*

($ii$) *the number of arbitrary parameters in the solution sets of equations* (2) *and* (1) *is* $n$*.*

We will denote the solution set of equation (1) when $A$ is nonderogatory by $\mathcal{U}_n$, that is

$$\mathcal{U}_n = \left\{ SBS^{-1}, B \in \mathcal{B}_n \right\}.$$

## 3. $n_k$-classes

Now we define a special set of matrices of order $n$, where the matrices are written with $k \leq n$ variables. We will study the close relation between these variables and the arbitrary parameters of the solutions sets of equations (1) and (2).

**Definition 3.1.** *Let $y_1, y_2, \ldots, y_k \in \mathbb{R}$ be variables and let $\alpha_{(ij)_l} \in \mathbb{R}$, $i, j = 1, \ldots, n$ and $l = 1, \ldots, k$ be arbitrary constants. The set of real matrices of order $n \geq k$*

$$\mathcal{V}_{n_k} = \left\{ \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{bmatrix}, v_{ij} = \sum_{l=1}^{k} \alpha_{(ij)_l} y_l \right\}$$

*is said to be an $n_k$-class of matrices.*

*If $k = n$ we say that it is an $n$-class and we denote by $\mathcal{V}_n$.*

It follows from definition (3.1) that $\mathcal{V}_{n_k}$ is an additive group and for $X \in \mathcal{V}_{n_k}$ we have $EX \in \mathcal{V}_{n_k}$, for any scalar matrix $E$.

We observe that in an $n_k$-class, if an element of a matrix is constant it has to be zero. We see this in the next examples.

*Example 3.2.* Let

$$\left\{ \begin{bmatrix} y_1 + 3y_2 & -y_1 \\ y_2 & \frac{1}{3}y_1 - 4y_2 \end{bmatrix}, y_1, y_2 \in \mathbb{R} \right\}$$

be a set. We have that this set is a 2-class, where $\alpha_{(11)_1} = 1$, $\alpha_{(11)_2} = 3$; $\alpha_{(12)_1} = -1$, $\alpha_{(12)_2} = 0$; $\alpha_{(21)_1} = 0$, $\alpha_{(21)_2} = 1$; $\alpha_{(22)_1} = \frac{1}{3}$, $\alpha_{(22)_2} = -4$. Also,

$$\left\{ \begin{bmatrix} y_1 + y_2 & -y_1 + y_3 & 0 \\ y_2 - y_1 & \frac{1}{4}y_1 - 4y_2 + y_3 & y_2 \\ -y_1 + y_2 - 5y_3 & -2y_1 + y_2 & 2y_1 + y_3 \end{bmatrix}, y_1, y_2, y_3 \in \mathbb{R} \right\}$$

is a 3-class, where $\alpha_{(13)_1} = \alpha_{(13)_2} = \alpha_{(13)_3} = 0$.

*Example 3.3.* Let

$$\left\{ \begin{bmatrix} y_1 + y_2 & 3 \\ y_2 & -y_1 \end{bmatrix}, y_1, y_2 \in \mathbb{R} \right\}$$

be a set. We have that this set is not a 2-class, considering that $v_{12}$ is a nonzero constant. Also,

$$\left\{ \begin{bmatrix} y_1 + y_2 & y_3 \\ y_4 & y_2 \end{bmatrix}, y_1, y_2, y_3, y_4 \in \mathbb{R} \right\}$$

is not a 2-class, because $k > n$.

*Example 3.4.* Let

$$\left\{\begin{bmatrix} 2y_1 & -3y_1 \\ y_1 & \frac{1}{2}y_1 \end{bmatrix}, y_1 \in \mathbb{R}\right\}$$

be a set. We have that this set is a $2_1$-class.

One of our principal objectives is to obtain conditions for the matrices of an $n$-class to be commutative. We begin by formalizing this definition.

**Definition 3.5.** *Let $\mathcal{V}_{n_k}$ be an $n_k$-class. If for any $A, B \in \mathcal{V}_{n_k}$, $AB = BA$ then, we say that $\mathcal{V}_{n_k}$ is a commutative $n_k$-class .*

*Example 3.6.* Let

$$\mathcal{V}_2 = \left\{\begin{bmatrix} y_1 & -2y_2 \\ y_2 & y_1 + 3y_2 \end{bmatrix}, y_1, y_2 \in \mathbb{R}\right\}.$$

Then, it can be verified that $\mathcal{V}_2$ is a commutative 2-class. On the other hand, the 2-class

$$\mathcal{V}_2' = \left\{\begin{bmatrix} y_1 & y_1 - y_2 \\ y_2 & y_1 - y_2 \end{bmatrix}, y_1, y_2 \in \mathbb{R}\right\}$$

is not a commutative 2-class.

*Example 3.7.* Let

$$\mathcal{V}_{3_2} = \left\{\begin{bmatrix} y_1 & -y_2 & y_1 + y_2 \\ -y_1 & y_2 & -y_1 - y_2 \\ y_2 & y_2 & 0 \end{bmatrix}, y_1, y_2 \in \mathbb{R}\right\}$$

Then, it can be verified that $\mathcal{V}_{3_2}$ is a commutative $3_2$-class.

We see next that if we multiply every matrix of an $n_k$-class by a constant matrix the resulting set is also an $n_k$-class:

**Lemma 3.8.** *Let $\mathcal{V}_{n_k}$ be an $n_k$-class and let $A$ be a real matrix. Then the sets*

$$\{AV, V \in \mathcal{V}_{n_k}\} \quad \textit{and} \quad \{VA, V \in \mathcal{V}_{n_k}\}$$

*are $n_k$-classes.*

*Proof.* Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

so for any matrix

$$V = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix}$$

an element of the product $AV$ can be written as

$$\begin{aligned} (AV)_{ij} &= \sum_{p=1}^{n} a_{ip}v_{pj} = \sum_{p=1}^{n} a_{ip}(\sum_{l=1}^{k} \alpha_{(pj)_l} y_l) = \\ &= \sum_{p=1}^{n}\sum_{l=1}^{k} a_{ip}\alpha_{(pj)_l} y_l = \sum_{l=1}^{k}\sum_{p=1}^{n} a_{ip}\alpha_{(pj)_l} y_l = \\ &= \sum_{l=1}^{k} \beta_{(ij)_l} y_l, \end{aligned}$$

where $\beta_{ij} = \sum_{p=1}^{n} a_{ip}\alpha_{(pj)}$ is a real constant, therefore this set is still an $n_k$-class and we denote it by $A\mathcal{V}_{n_k}$. In similar way we get $\mathcal{V}_{n_k}A$. ■

Next we consider the cases $k = n$ and $k < n$ separately.

### 3.1 Case $k = n$

**Proposition 3.9.** *Let $A$ be a real matrix, $A$ is nonderogatory if and only if the solution set of the equation* (1) *is a commutative $n$-class.*

*Proof.* (⇒) Let $\mathcal{U}_n$ be the the solution set of the equation (1), $A$ is nonderogatory, so by Corollary 2.12 (*ii*) the number of arbitrary parameters is $n$ and then $\mathcal{U}_n$ has $n$ variables $x_1, x_2, \dots, x_n$ and so, it is an $n$-class.

Let $A = SJ_AS^{-1}$ where $J_A$ is the Jordan canonical form of $A$. Then any two solutions of the equation (1) can be written as $X = SBS^{-1}$ and $X' = SB'S^{-1}$, where $B$ and $B'$ are solutions of the equation (2).

Now, by Corollary 2.12 (*i*) $B$ and $B'$ are block diagonal matrices with the blocks being upper triangular Toeplitz matrices of the same order, so $BB' = B'B$ by Proposition 2.2 and thus,

$$XX' = SBS^{-1}SB'S^{-1} = SBB'S^{-1} = SB'BS^{-1} = SB'S^{-1}SBS^{-1} = X'X,$$

hence $\mathcal{U}_n$ is a commutative $n$-class.

(⇐) By definition of solution set all matrices of $\mathcal{U}_n$ commute, so we have only to use that $\mathcal{U}_n$ is an $n$-class. This is done by contrapositive. We suppose that

$A$ is derogatory, then by Corollary 2.11 (*ii*) the solution set $\mathcal{U}_n$ of (1) has more then $n$ arbitrary parameters and therefore $\mathcal{U}_n$ has more than $n$ variables and so it is not an $n$-class. ∎

*Example 3.10.* Let

$$A = \begin{bmatrix} -3 & -1 & -3 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix},$$

whose Jordan form and similarity matrix are

$$J_A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

respectively. Solving the equation (2) we obtain the following solution set

$$\left\{ Y = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & x_3 \\ 0 & 0 & x_2 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Hence,

$$\left\{ X = SYS^{-1} = \begin{bmatrix} x_1 - x_3 & -x_3 & x_1 - x_2 \\ -x_1 + x_2 + x_3 & x_2 + x_3 & -x_1 + x_2 \\ x_3 & x_3 & x_2 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Finally, considering that $A$ is nonderogatory, we have from Proposition 3.9 that this set is a commutative 3-class.

**Corollary 3.11.** *The solution set of equation* (2) $\mathcal{B}_n$ *is a commutative $n$-class.*

*Proof.* Using Proposition 2.2, it is a direct verification. ∎

**Corollary 3.12.** *If $\mathcal{U}_n$ is a commutative $n$-class then $\mathcal{U}_n$ is a commutative ring with unity.*

*Proof.* An $n$-class is an additive group, so we have only to show that $I \in \mathcal{U}_n$ and that $\mathcal{U}_n$ is closed under multiplication.

If we write $B = I$ in Corollary 2.12 (*i*) it is immediate that $I \in \mathcal{U}_n$.

Now, we consider $X = SBS^{-1}$ and $X' = SB'S^{-1}$, so $XX' = SBS^{-1}SB'S^{-1} = SBB'S^{-1}$ where $BB'$ are block diagonal with the blocks being upper triangular Toeplitz matrices by Proposition 2.2, hence $XX' \in \mathcal{U}_n$. ∎

We observe that $\mathcal{U}_n$ and $\mathcal{B}_n$ are $n$-classes and that

$$\mathcal{U}_n = \left\{SBS^{-1}, B \in \mathcal{B}_n\right\},$$

so we can write $\mathcal{U}_n = S\mathcal{B}_n S^{-1}$ using the Lemma 3.8 twice.

In the next example we can see that a commutative $n$-class can be written in different ways:

*Example 3.13.* Let

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix},$$

we can verify that $A$ is a nonderogatory matrix. We can compute a solution

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

of (1), with $x_3$ and $x_4$ as arbitrary parameters and we obtain the following commutative 2-class,

$$\mathcal{U}_2 = \left\{ \begin{bmatrix} -x_3 + x_4 & -4x_3 \\ x_3 & x_4 \end{bmatrix}, x_3, x_4 \in \mathbb{R} \right\}.$$

On the other hand, if we choose $x_1$ and $x_4$ as arbitrary parameters, we get

$$\mathcal{U}_2' = \left\{ \begin{bmatrix} x_1 & 4(x_1 - x_4) \\ x_4 - x_1 & x_4 \end{bmatrix}, x_1, x_4 \in \mathbb{R} \right\}.$$

Finally, we can verify that $\mathcal{U}_2$ and $\mathcal{U}_2'$ are the same commutative 2-class by substituting $x_1 = -x_3 + x_4$ in $\mathcal{U}_2'$.

## 3.2 Case $k < n$

**Proposition 3.14.** *Let $\mathcal{V}_{n_k}$ be an $n_k$-class. If $k = 1$ then, $\mathcal{V}_{n_k}$ is commutative.*

*Proof.* $\mathcal{V}_{n_1}$ is an $n_1$-class, so we can write

$$\mathcal{V}_{n_1} = \left\{ \begin{bmatrix} \alpha_{11}y_1 & \dots & \alpha_{1n}y_1 \\ \vdots & \ddots & \vdots \\ \alpha_{n1}y_1 & \dots & \alpha_{nn}y_1 \end{bmatrix}, y_1 \in \mathbb{R} \right\}$$

Let $A, B \in \mathcal{V}_{n_1}$ then,

$$A = \begin{bmatrix} y_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & y_1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{bmatrix}, y_1 \in \mathbb{R}$$

and

$$B = \begin{bmatrix} y_2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & y_2 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{bmatrix}, y_2 \in \mathbb{R}$$

so $AB = BA$. ∎

**Definition 3.15.** *An $n_k$-class $\mathcal{V}_{n_k}$ is said to be scalar if any matrix $A \in \mathcal{V}_{n_k}$ is scalar.*

**Definition 3.16.** *An $n_k$-class $\mathcal{V}_{n_k}$ is said to be singular if any matrix $A \in \mathcal{V}_{n_k}$ is singular.*

**Definition 3.17.** *An $n_k$-class $\mathcal{V}_{n_k}$ is said to be derogatory if any matrix $A \in \mathcal{V}_{n_k}$ is derogatory.*

**Definition 3.18.** *An $n_k$-class $\mathcal{V}_{n_k}$ is said to be nilpotent if any matrix $A \in \mathcal{V}_{n_k}$ is nilpotent.*

We recall the following elementary facts:
- in a singular matrix, at least one of its eigenvalues is zero,
- in a derogatory matrix at least two of its eigenvalues are equal,
- in a nilpotent matrix all its eigenvalues are zero.

We will use this later.

We saw in Corollary 3.12 that any commutative $n$-class $\mathcal{U}_n$ is a ring. However not all commutative $n_k$-classes with $k < n$, are rings, because they could be not closed under usual product of matrices, this can be verified in the following example:

*Example 3.19.* The commutative $3_1$-class

$$\mathcal{U}'_{3_1} = \left\{ \begin{bmatrix} y_1 & 0 & 0 \\ -y_1 & 0 & y_1 \\ 0 & 0 & y_1 \end{bmatrix}, y_1 \in \mathbb{R} \right\}$$

is a ring, but the commutative $3_2$-class

$$\mathcal{U}_{3_2} = \left\{ \begin{bmatrix} y_1 & -y_1 + y_2 & -2y_1 + 2y_2 \\ 3y_1 - 3y_2 & y_2 & 7y_1 - 7y_2 \\ 0 & \frac{1}{3}(5y_1 - 5y_2) & \frac{1}{3}(-8y_1 + 11y_2) \end{bmatrix}, y_1, y_2 \in \mathbb{R} \right\}$$

is not a ring.

Next, we present a necessary and sufficient condition for an $n_k$-class be a ring, when $n = 2$ and $k = 1$.

**Proposition 3.20.** *A commutative $2_1$-class $\mathcal{V}_{2_1}$ is a ring if and only if it is a singular $2_1$-class or it is a scalar $2_1$-class .*

*Proof.* ($\Rightarrow$) By hypothesis

$$\mathcal{V}_{2_1} = \left\{ \begin{bmatrix} ax & bx \\ cx & dx \end{bmatrix}, x \in \mathbb{R} \right\}$$

is a $2_1$-class closed under the usual product of matrices. Let

$$A = \begin{bmatrix} ax & bx \\ cx & dx \end{bmatrix}, B = \begin{bmatrix} ay & by \\ cy & dy \end{bmatrix} \in \mathcal{V}_{2_1}$$

then

$$AB = \begin{bmatrix} (a^2+bc)xy & b(a+d)xy \\ c(a+d)xy & (d^2+bc)xy \end{bmatrix}.$$

Considering that $AB \in \mathcal{V}_{2_1}$, we have

$$\begin{cases} a^2 + bc = a \\ b(a+d) = b \\ c(a+d) = c \\ d^2 + bc = d \end{cases},$$

so:

**i)** If $b \neq 0$ then $a = 1 - d$ and $c = \frac{d(1-d)}{b}$. Thus, we have a singular $2_1$-class,

$$\left\{ \begin{bmatrix} (1-d)x & bx \\ \frac{d(1-d)}{b}x & dx \end{bmatrix}, x \in \mathbb{R} \right\}$$

**ii)** If $c \neq 0$ then $a = 1 - d$ and $b = \frac{d(1-d)}{c}$. We obtain the singular $2_1$-class

$$\left\{ \begin{bmatrix} (1-d)x & \frac{d(1-d)}{c}x \\ cx & dx \end{bmatrix}, x \in \mathbb{R} \right\}$$

**iii)** If $b = 0$ and $c = 0$ then $a = 1$, $d = 1$. We obtain the scalar $2_1$-class

$$\left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, x \in \mathbb{R} \right\}$$

**iv)** If $a = 0$ then $c = 0$ (or $b = 0$) and $d = 1$. We obtain the singular $2_1$-class

$$\left\{ \begin{bmatrix} 0 & 0 \\ cx & x \end{bmatrix}, x \in \mathbb{R} \right\}$$

or

$$\left\{ \begin{bmatrix} 0 & bx \\ 0 & x \end{bmatrix}, x \in \mathbb{R} \right\}.$$

($\Leftarrow$) Let $\mathcal{V}_{2_1}$ be a singular $2_1$-class. Then, $\mathcal{V}_{2_1}$ has one of the following forms:

$$\mathcal{V}'_{2_1} = \left\{ \begin{bmatrix} \alpha bx & bx \\ \alpha dx & dx \end{bmatrix}, x \in \mathbb{R} \right\}$$

or

$$\mathcal{V}''_{2_1} = \left\{ \begin{bmatrix} \alpha cx & \alpha dx \\ cx & dx \end{bmatrix}, x \in \mathbb{R} \right\}$$

where $\alpha$ is a real constant.

Let $A = \begin{bmatrix} \alpha bx & bx \\ \alpha dx & dx \end{bmatrix}$ and $B = \begin{bmatrix} \alpha by & by \\ \alpha dy & dy \end{bmatrix}$ matrices of $\mathcal{V}'_{2_1}$.

Then $AB = \begin{bmatrix} \alpha b(\alpha b + d)xy & b(\alpha b + d)xy \\ \alpha d(\alpha b + d)xy & d(\alpha b + d)xy \end{bmatrix} \in \mathcal{V}'_{2_1}$ and so $\mathcal{V}'_{2_1}$ is closed under the usual product of matrices, and so it is a ring.

In a similar way we have that $\mathcal{V}''_{2_1}$ is closed under the usual product of matrices.

Furthermore, it is obvious that a scalar $2_1$-class is also closed under the usual product of matrices. ■

If $k = 1$ and $n > 2$, the Proposition 3.20 is not true. We can see this in the following example:

*Example 3.21.* Let

$$\mathcal{V}_{3_1} = \left\{ \begin{bmatrix} y_1 & 0 & y_1 \\ 0 & y_1 & 0 \\ y_1 & 0 & y_1 \end{bmatrix}, y_1 \in \mathbb{R} \right\}$$

be a commutative singular $3_1$-class. $\mathcal{V}_{3_1}$ is not closed under the usual product of matrices and therefore it is not a ring.

From now on, we will only consider commutative $n$-classes $\mathcal{U}_n$ in the conditions of Proposition 3.9, that is, commutative $n$-classes that are a solution set of an equation of the form (1) with $A$ being nonderogatory.

We recall that eigenvalues are preserved by similarity transformation, then the distinct eigenvalues of the matrices of $\mathcal{B}_n$ are eigenvalues of the respective matrices of $\mathcal{U}_n$ with the same partial multiplicities.

Now, we will construct $n_k$-classes $\mathcal{U}_{n_k}$, as being subsets of $\mathcal{U}_n$ by applying one of the following transformations *types*:

*1- Make one or more eigenvalues of the matrices of* $\mathcal{U}_n$ *equal to zero;*

*2- Make one or more eigenvalues of the matrices of* $\mathcal{U}_n$ *be equal;*

We remark that an $n_k$-class obtained by a transformation of *type 1* is singular and an $n_k$-class obtained by a transformation of *type 2* is derogatory. Furthermore, if we make all eigenvalues of the matrices of $\mathcal{U}_n$ being equal to zero we obtain a nilpotent $n_k$-class.

Besides that, any $n_k$-class $\mathcal{U}_{n_k} \subset \mathcal{U}_n$ obtained by one of these two transformations is commutative because, it is a subset of a commutative set.

Next, we show that any $n_k$-class $\mathcal{U}_{n_k} \subset \mathcal{U}_n$ obtained by one of these transformations is a ring.

**Proposition 3.22.** *Let $\mathcal{U}_n$ be a commutative $n$-class and $\mathcal{U}_{n_k} \subset \mathcal{U}_n$ an $n_k$-class obtained by a transformations of type 1. Then $\mathcal{U}_{n_k}$ is a sub-ring of $\mathcal{U}_n$.*

*Proof.* An $n$-class $\mathcal{U}_n$ obtained from the equation (1) have the form $\mathcal{U}_n = S\mathcal{B}_nS^{-1}$, where $\mathcal{B}_n$ is the solution set of the equation (2) and by Corollary 2.12 any matrix $B \in \mathcal{B}_n$ is block diagonal in which the blocks in diagonal $B_{n_i} \in \mathcal{T}_n$, $i = 1, \dots, p$. Furthermore, the element $x_i$ of the diagonal of $B_{n_i}$ is an eigenvalue of $B$ and therefore it is an eigenvalue of $SBS^{-1} \in \mathcal{U}_n$. So, we make one eigenvalue, say the first for simplicity, $x_1 = 0$, such that $k = n - 1$. Hence, we consider the set

$$\mathcal{C}_n = \{B \in \mathcal{B}_n, x_1 = 0\} \subset \mathcal{B}_n,$$

which is an $n_{n-1}$-class and for $C_1, C_2 \in \mathcal{C}_n$ we have that $C_1C_2 \in \mathcal{C}_n$ by Proposition 2.5 since $C_1, C_2 \in \mathcal{K}_n$.

Now, using Lemma 3.8 we have

$$\mathcal{U}_{n_k} = S\mathcal{C}_nS^{-1}, k = n - 1.$$

It is obvious that $\mathcal{U}_{n_k} \subset \mathcal{U}_n$ and to prove that $\mathcal{U}_{n_k}$ is a sub-ring we have only to verify that $\mathcal{U}_{n_k}$ is closed for the usual product of matrices, that is, let $A_1, A_2 \in \mathcal{U}_{n_k}$, then $A_1 = SC_1S^{-1}$ and $A_2 = SC_2S^{-1}$ for $C_1, C_2 \in \mathcal{C}_n$.

Thus

$$A_1A_2 = SC_1S^{-1}SC_2S^{-1} = SC_1C_2S^{-1} \in \mathcal{U}_{n_k}$$

since $C_1C_2 \in \mathcal{C}_n$.

This is still valid if we consider more than one eigenvalue equal to zero, that is for any $0 < k < n - 1$. ∎

**Proposition 3.23.** *Let $\mathcal{U}_n$ be a commutative $n$-class and $\mathcal{U}_{n_k} \subset \mathcal{U}_n$ an $n_k$-class obtained by a transformations of type 2. Then $\mathcal{U}_{n_k}$ is a sub-ring of $\mathcal{U}_n$.*

*Proof.* It is similar to the proof of Proposition 3.22 by making $x_1 = x_2$, where $x_1$ and $x_2$ are the diagonal elements of the blocks $B_{n_1}$ and $B_{n_2}$ respectively. ∎

**Corollary 3.24.** *If $\mathcal{U}_{n_k}$ is obtained by a transformation of type 2, that is $\mathcal{U}_{n_k}$ is derogatory, then $\mathcal{U}_{n_k}$ has unity.*

*Proof.* $A \in \mathcal{U}_{n_k}$, so we can write $A = SKS^{-1}$, where $K \in \mathcal{K}_n$ and the blocks $K_{n_i} \in \mathcal{T}_n^+$, we take $K_{n_i} = I_{n_i}$ and thus $\mathcal{U}_{n_k}$ has unity. ∎

*Example 3.25.* Let

$$A = \begin{bmatrix} -3 & -1 & -3 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix},$$

whose Jordan canonical form and similarity matrix are

$$J_A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

respectively. Solving the equation (2) we obtain the following solution set

$$\mathcal{B}_3 = \left\{ \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & x_3 \\ 0 & 0 & x_2 \end{bmatrix} x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Hence,

$$\mathcal{U}_3 = S\mathcal{B}_3S^{-1} = \left\{ \begin{bmatrix} x_1 + x_3 & x_1 - x_2 & -x_1 + x_2 + x_3 \\ -x_1 + x_2 - 2x_3 & -x_1 + 2x_2 & x_1 - x_2 - 2x_3 \\ -x_1 + x_2 - x_3 & -x_1 + x_2 & x_1 - x_3 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\}$$

is the solution set of the equation (1), and considering that $A$ is nonderogatory, by Proposition 3.9 $\mathcal{U}_3$ is a commutative 3-class. The eigenvalues of the matrices of $\mathcal{B}_3$ and consequently the eigenvalues of the matrices of $\mathcal{U}_3$ are $x_1$ and $x_2$. Now, we consider the following transformations:

**a)** If we make $x_1 = 0$ (an eigenvalue) in $\mathcal{B}_3$ we get

$$\mathcal{C}_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_2 & x_3 \\ 0 & 0 & x_2 \end{bmatrix}, x_2, x_3 \in \mathbb{R} \right\}$$

and so we obtain the commutative singular $3_2$-class

$$\mathcal{U}_{3_2} = S\mathcal{C}_2S^{-1} = \left\{ \begin{bmatrix} x_3 & -x_2 & x_2 + x_3 \\ x_2 - 2x_3 & 2x_2 & -x_2 - 2x_3 \\ x_2 - x_3 & x_2 & -x_3 \end{bmatrix}, x_2, x_3 \in \mathbb{R} \right\},$$

which is a sub-ring of $\mathcal{U}_3$.

**b)** If we make $x_2 = 0$ in $\mathcal{B}_3$ we get

$$\mathcal{C}'_2 = \left\{ \begin{bmatrix} x_1 & 0 & 0 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix}, x_1, x_3 \in \mathbb{R} \right\}$$

and so we obtain the commutative singular $3_2$-class

$$\mathcal{U}'_{3_2} = S\mathcal{C}'_2S^{-1} = \left\{ \begin{bmatrix} x_1 + x_3 & x_1 & -x_1 + x_3 \\ -x_1 - 2x_3 & -x_1 & x_1 - 2x_3 \\ -x_1 - x_3 & -x_1 & x_1 - x_3 \end{bmatrix}, x_1, x_3 \in \mathbb{R} \right\},$$

which is a sub-ring of $\mathcal{U}_3$.

**c)** If in $\mathcal{B}_3$ we make $x_2 = 0$ and $x_1 = 0$ we get

$$\mathcal{C}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix}, x_3 \in \mathbb{R} \right\}$$

and so we obtain the commutative nilpotent $3_1$-class

$$\mathcal{U}_{3_1} = S\mathcal{C}_1 S^{-1} = \left\{ \begin{bmatrix} x_3 & 0 & x_3 \\ -2x_3 & 0 & -2x_3 \\ -x_3 & 0 & -x_3 \end{bmatrix}, x_3 \in \mathbb{R} \right\},$$

which is a sub-ring of $\mathcal{U}_3$.

**d)** Finally if in $\mathcal{B}_3$ we make $x_1 = x_2$, then we get

$$\mathcal{C''}_2 = \left\{ \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_1 & x_3 \\ 0 & 0 & x_1 \end{bmatrix}, x_1, x_3 \in \mathbb{R} \right\}$$

and we obtain the commutative derogatory $3_2$-class

$$\mathcal{U}''_{3_2} = S\mathcal{C''}_2 S^{-1} = \left\{ \begin{bmatrix} x_1 + x_3 & 0 & x_3 \\ -2x_3 & x_1 & -2x_3 \\ -x_3 & 0 & x_1 - x_3 \end{bmatrix}, x_1, x_3 \in \mathbb{R} \right\},$$

which is a sub-ring of $\mathcal{U}_3$.

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